NOTE ON CHAPTER 26 OF DAVENPORT'S MULTIPLICATIVE NUMBER THEORY

STEVE FAN

Abstract. In Chapter 26 of Davenport's classic book [4], it is shown that every sufficiently large odd positive integer can be written as the sum of three primes. In this short note we explain how the method used there can be modified to show that there are infinitely many 3-term arithmetic progressions in the sequence of primes. The note resulted from the author's independent reading of [4, Ch. 26].

In [4, Ch. 26] it is shown that every sufficiently large odd positive integer can be written as the sum of three primes. This is a consequence of Vinogradov's asymptotic formula for

$$r(N) := \sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3),$$

where N is any positive integer. The formula states

$$r(N) = \frac{1}{2}\mathfrak{S}(N)N^2 + O(N^2(\log N)^{-A}),$$

where $A > 0$ is arbitrary but fixed, and $\mathfrak{S}(N)$ is the singular series for the odd Goldbach conjecture defined by

$$\mathfrak{S}(N) := \prod_{p \mid N}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \nmid N}\left(1 + \frac{1}{(p-1)^3}\right).$$

In the present note we explain how the method used there can be modified with no difficulty to show that there are infinitely many (nontrivial) 3-term arithmetic progressions in the sequence of primes. This was first proved in 1939 by van der Corput [3], which follows immediately from the following theorem.

Theorem. *For every positive integer N, let*

$$f(N) := \sum_{\substack{n_1,n_2,n_3 \le N \\ n_1+n_3=2n_2}} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3).$$

Then for any fixed $A > 0$ we have

$$f(N) = \mathfrak{S}N^2 + O(N^2(\log N)^{-A}),$$

where

$$\mathfrak{S} := \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right).$$

We introduce the variable N in the theorem to impose restrictions on the sizes of the unknowns n_1, n_2, n_3 of the equation $n_1 + n_3 = 2n_2$. Note also that $\mathfrak{S}$ is now a positive number independent of N. From the above theorem we easily derive the following result as a corollary.

Corollary. *As $N\to\infty$, the number of 3-term arithmetic progressions in primes up to N is*
$$\frac{1}{2}(\mathfrak{S}+o(1))\frac{N^2}{(\log N)^3}.$$

Proof. Let
$$A_N:=\{(n_1,n_2,n_3)\in\mathbb{Z}^3\cap[1,N]^3\colon n_1,n_2,n_3 \text{ are prime powers and } n_1+n_2=2n_3\}$$
and $a_N:=\#A_N$. It is clear that $f(N)\le a_N(\log N)^3$. By Theorem we obtain
$$\varliminf_{N\to\infty}\frac{a_N}{N^2(\log N)^{-3}}\ge\mathfrak{S}.$$
Let $\delta\in(0,1)$ be an arbitrary positive real number. Then it follows from Theorem that
$$\begin{aligned}a_N&=\sum_{\substack{(n_1,n_2,n_3)\in A_N\\ n_i\le N^\delta \text{ for some } i}}1+\sum_{\substack{(n_1,n_2,n_3)\in A_N\\ n_1,n_2,n_3>N^\delta}}1\\&\le 3N^{1+\delta}+(\log N^\delta)^{-3}\sum_{\substack{(n_1,n_2,n_3)\in A_N\\ n_1,n_2,n_3>N^\delta}}\Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\\&\le\delta^{-3}\mathfrak{S}N^2(\log N)^{-3}+O_\delta(N^2(\log N)^{-4}).\end{aligned}$$
This implies
$$\varlimsup_{N\to\infty}\frac{a_N}{N^2(\log N)^{-3}}\le\delta^{-3}\mathfrak{S}.$$
Since $\delta\in(0,1)$ is arbitrary, we have
$$\varlimsup_{N\to\infty}\frac{a_N}{N^2(\log N)^{-3}}\le\mathfrak{S}.$$
We have thus proved
$$a_N=(\mathfrak{S}+o(1))\frac{N^2}{(\log N)^3}.$$
Note that the contribution to a_N from the triples $(n_1,n_2,n_3)\in A_N$ such that not all n_1,n_2,n_3 are prime is at most
$$3N\sum_{\substack{p^v\le N\\ v\ge 2}}1\le\frac{3N}{\log 2}\sum_{\substack{p^v\le N\\ v\ge 2}}\log p\le\frac{3N\log N}{\log 2}\pi(\sqrt{N})\ll N^{3/2}$$
by Chebyshev's estimate [10, Theorem 7]. Moreover, the contribution to a_N from the triples $(p,p,p)\in A_N$ for some prime $p\le N$ is
$$\pi(N)\ll\frac{N}{\log N}.$$
Hence the number of 3-term arithmetic progressions in primes up to N is
$$\frac{1}{2}\left(a_N+O(N^{3/2})+O\left(\frac{N}{\log N}\right)\right)=\frac{1}{2}(\mathfrak{S}+o(1))\frac{N^2}{(\log N)^3}.$$
This completes the proof. $\square$

More generally, one can study k-term arithmetic progressions in primes for positive integers $k \geq 2$. One may ask whether there are infinitely many k-term arithmetic progressions in the sequence of primes. The case $k = 2$ is captured by the prime number theorem [10, Theorem 6] and the case $k = 3$ was resolved by van der Corput [3] as mentioned above. It is interesting to compare this problem to Szemerédi's theorem and conjecture that primes do contain arbitrarily long arithmetic progressions. Szemerédi's theorem, conjectured by Erdős and Turán [5] in 1936 and proved in full generality by Szemerédi [12] in 1975, asserts that any subset A of positive integers with positive upper density

$$\varlimsup_{N\to\infty} \frac{\#(A\cap[1,N])}{N} > 0$$

must contain arbitrarily long arithmetic progressions. Unfortunately, since the sequence of primes has natural density equal to 0, Szemerédi's theorem does not apply at once. Major progress was made by Green and Tao [6] in 2008 who proved that for every positive integer $k \geq 2$, the primes contain infinitely many k-term arithmetic progressions. In fact, they showed that any subset A of primes with positive relative upper density

$$\varlimsup_{N\to\infty} \frac{\#(A\cap[1,N])}{\pi(N)} > 0$$

must contain infinitely many k-term arithmetic progressions for every $k \geq 2$. This is the celebrated Green-Tao theorem, the proof of which uses an extension of Szemerédi's theorem to subsets of pseudorandom integers and depends heavily on deep machinery from ergodic theory. It is now known [7, 8, 9] that for every $k \geq 2$, the number of k-term arithmetic progressions in primes up to N is

$$(\mathfrak{S}_k + o(1))\frac{N^2}{(\log N)^k},$$

where

$$\mathfrak{S}_k := \frac{1}{2(k-1)} \prod_{p\leq k} \frac{1}{p}\left(\frac{p}{p-1}\right)^{k-1} \prod_{p>k}\left(1-\frac{k-1}{p}\right)\left(\frac{p}{p-1}\right)^{k-1}.$$

Another conjecture of Erdős and Turán [5] states that any subset A of positive integers with

$$\sum_{n\in A}\frac{1}{n} = \infty$$

contains arbitrarily long arithmetic progressions. It is well known [10, Theorem 19] that $\sum_p 1/p = \infty$. Thus the conjecture of Erdős and Turán, if true, would imply at once that the primes contain arbitrarily long arithmetic progressions. In fact, it is not hard to see that this conjecture, if true, would include both Szemerédi's theorem and the Green-Tao theorem as special cases. For instance, suppose that A is a subset of positive integers with positive upper density. Then there exist a constant $c \in (0, 1/2)$ and a strictly increasing sequence $\{N_i\}_{i=1}^{\infty}$ of positive integers such that $N_{i+1} \geq 2N_i$ and $A(N_i) > 2cN_i$ for all $i \geq 1$, where $A(x) := \#(A \cap [1, x])$ for all $x \geq 1$. For any $x \in [N_i, 2N_i]$ we have

$$\frac{A(x)}{x} \geq \frac{A(N_i)}{2N_i} > c.$$

It follows by partial summation that

$$\sum_{n\in A\cap[1,2N_m]}\frac{1}{n}\geq\int_1^{2N_m}\frac{A(x)}{x^2}\,dx\geq c\sum_{i=1}^{m}\int_{N_i}^{2N_i}\frac{1}{x}\,dx=mc\log 2\to\infty$$

as $m\to\infty$. This shows that the Erdős-Turán conjecture implies Szemerédi's theorem. By a similar argument together with Chebyshev's estimate [10, Theorem 7], one can prove that the Erdős-Turán conjecture also implies the Green-Tao theorem. Unfortunately, the conjecture does not apply to the set of twin primes, since Brun [2] showed that the sum of the reciprocals of the twin primes converges. The Erdős-Turán conjecture is currently open.[1]

Now we describe how to prove Theorem. Like that of estimating $r(N)$, the problem of estimating $f(N)$ is of complexity one from the point of view of higher order Fourier analysis, meaning in particular that the circle method will often work effectively. As we shall see, only modest changes need to be made. The starting point is the observation that

$$f(N)=\int_0^1 S(\alpha)^2S(-2\alpha)\,d\alpha,$$

where

$$S(\alpha):=\sum_{n\leq N}\Lambda(n)e(n\alpha)$$

with $e(x):=e^{2\pi ix}$ for any $x\in\mathbb{R}$. Put $P:=(\log N)^B$ and $Q:=N(\log N)^{-B}$, where $B:=2A+10$. To employ the circle method, we define a typical major arc $\mathfrak{M}(q,a)$ centered at $a/q\in\mathbb{Q}$, where $q\leq P$ and $1\leq a\leq q$ with $\gcd(a,q)=1$, by

$$\mathfrak{M}(q,a):=\left\{\alpha\in S^1\colon\ \left\|\alpha-\frac{a}{q}\right\|\leq\frac{1}{Q}\right\},$$

where $S^1:=\mathbb{R}/\mathbb{Z}$ denotes the unit circle and $\|x\|:=\min_{n\in\mathbb{Z}}|x-n|$. As usual, let $\mathfrak{M}$ be the union of these major arcs and let $\mathfrak{m}:=S^1\setminus\mathfrak{M}$.

Consider now the major arc $\mathfrak{M}(q,a)$. For $\alpha\in\mathfrak{M}(q,a)$ we write $\alpha=a/q+\beta$. It is proved in [4, Ch. 26] that

$$S(\alpha)=\frac{\mu(q)}{\varphi(q)}T(\beta)+O(N\exp(-c_1\sqrt{\log N}))\tag{1}$$

for some constant $c_1=c_1(B)>0$ depending only on B, where

$$T(\beta):=\sum_{n\leq N}e(n\alpha).$$

It follows that

$$S(\alpha)^2=\frac{\mu(q)^2}{\varphi(q)^2}T(\beta)^2+O(N^2\exp(-c_1\sqrt{\log N})).$$

If $2\nmid q$, then we have similarly

$$S(-2\alpha)=\frac{\mu(q)}{\varphi(q)}T(-2\beta)+O(N\exp(-c_1\sqrt{\log N})).$$

[1] The case $k=3$ has recently been settled by Bloom and Sisask [1] as a consequence of their estimate that $r_3(N)\leq N/(\log N)^{1+c}$ for some suitable $c>0$, improving Roth's result [11] that $r_3(N)\ll N/\log\log N$. Here $r_3(N)$ is the size of the largest subset of $\{1,...,N\}$ containing no 3-term arithmetic progressions.

Thus we have

$$S(\alpha)^2S(-2\alpha)=\frac{\mu(q)}{\varphi(q)^3}T(\beta)^2T(-2\beta)+O(N^3\exp(-c_1\sqrt{\log N})).$$

On the other hand, if $2\mid q$, then we replace q by $q/2$ in the derivation of (1) to get

$$S(-2\alpha)=\frac{\mu(q/2)}{\varphi(q/2)}T(-2\beta)+O(N\exp(-c_1\sqrt{\log N})).$$

It follows that

$$\begin{aligned}S(\alpha)^2S(-2\alpha)&=\frac{\mu(q)^2\mu(q/2)}{\varphi(q)^2\varphi(q/2)}T(\beta)^2T(-2\beta)+O(N^3\exp(-c_1\sqrt{\log N}))\\&=-\frac{\mu(q)}{\varphi(q)^3}T(\beta)^2T(-2\beta)+O(N^3\exp(-c_1\sqrt{\log N})).\end{aligned}$$

Hence the contribution of $\mathfrak{M}(q,a)$ to $f(N)$ is

$$(-1)^{q-1}\frac{\mu(q)}{\varphi(q)^3}\int_{-1/Q}^{1/Q}T(\beta)^2T(-2\beta)\,d\beta+O(N^2\exp(-c_2\sqrt{\log N})),$$

where $c_2=c_2(B)>0$ depends only on B. Therefore, the contribution of $\mathfrak{M}$ to $f(N)$ is

$$\sum_{q\le P}(-1)^{q-1}\frac{\mu(q)}{\varphi(q)^2}\int_{-1/Q}^{1/Q}T(\beta)^2T(-2\beta)\,d\beta+O(N^2\exp(-c_3\sqrt{\log N})),\tag{2}$$

where $c_3=c_3(B)>0$ depends only on B. Following [4, Ch. 26] we see that

$$\begin{aligned}\int_{-1/Q}^{1/Q}T(\beta)^2T(-2\beta)\,d\beta&=\int_0^{1/Q}T(\beta)^2T(-2\beta)\,d\beta+\int_{1-1/Q}^{1}T(\beta)^2T(-2\beta)\,d\beta\\&=\int_0^{1}T(\beta)^2T(-2\beta)\,d\beta-\int_{1/Q}^{1-1/Q}T(\beta)^2T(-2\beta)\,d\beta\end{aligned}$$

and

$$\left|\int_{1/Q}^{1-1/Q}T(\beta)^2T(-2\beta)\,d\beta\right|\le N\int_{1/Q}^{1-1/Q}\|\beta\|^{-2}\,d\beta=2N\int_{1/Q}^{1/2}\beta^{-2}\,d\beta\ll N^2(\log N)^{-B}.$$

Observe that

$$\begin{aligned}\int_0^1T(\beta)^2T(-2\beta)\,d\beta&=\#\{(n_1,n_2,n_3)\in\mathbb{Z}^3\cap[1,N]^3\colon n_1+n_2=2n_3\}\\&=\left\lfloor\frac{N}{2}\right\rfloor^2+\left\lfloor\frac{N+1}{2}\right\rfloor^2\\&=\frac{N^2}{2}+O(N),\end{aligned}$$

where $\lfloor x\rfloor$ denotes the integer part of x for any $x\in\mathbb{R}$. It follows that

$$\int_{-1/Q}^{1/Q}T(\beta)^2T(-2\beta)\,d\beta=\frac{N^2}{2}+O(N^2(\log N)^{-B}).\tag{3}$$

A well-known result [10, Theorem 327] states

$$\lim_{n\to\infty}\frac{\varphi(n)}{n^{1-\delta}}=\infty$$

for any given $\delta>0$. To see this, note that φ is multiplicative and that

$$\frac{\varphi(p^m)}{p^{m(1-\delta)}}=p^{m\delta}\left(1-\frac{1}{p}\right)\geq\frac{1}{2}p^{m\delta}\to\infty$$

as $p^m\to\infty$. Taking $\delta=1/(2B)$ we obtain

$$\sum_{q>P}\frac{1}{\varphi(q)^2}\ll\sum_{q>P}\frac{1}{q^{2(1-\delta)}}\ll P^{-1+2\delta}=(\log N)^{-B+1}.$$

Hence the series

$$\sum_{q=1}^{\infty}(-1)^{q-1}\frac{\mu(q)}{\varphi(q)^2}$$

is absolutely convergent. It has the infinite product expansion

$$\prod_p\left(1+(-1)^{p-1}\frac{\mu(p)}{\varphi(p)^2}\right)=2\mathfrak{S}.$$

Thus we have

$$\sum_{q\leq P}(-1)^{q-1}\frac{\mu(q)}{\varphi(q)^2}=2\mathfrak{S}+O((\log N)^{-B+1}).$$

Combining this with (2) and (3) we see that the contribution of $\mathfrak{M}$ to $f(N)$ is

$$\int_{\mathfrak{M}}S(\alpha)^2S(-2\alpha)\,d\alpha=\mathfrak{S}N^2+O(N^2(\log N)^{-B+1}).\tag{4}$$

Now we consider the contribution of $\mathfrak{m}$ to $f(N)$. Let $\alpha\in\mathfrak{m}$. By Dirichlet's theorem on Diophantine approximation, there exists $a/q\in\mathbb{Q}$ with $q\leq P$ and $\gcd(a,q)=1$ such that $\|\alpha-a/q\|\leq 1/qQ$. Then

$$\left\|-2\alpha+\frac{2a}{q}\right\|=\min_{n\in\mathbb{Z}}\left|-2\alpha+\frac{2a}{q}+n\right|\leq\min_{n\in 2\mathbb{Z}}\left|-2\alpha+\frac{2a}{q}+n\right|=2\left\|\alpha-\frac{a}{q}\right\|\leq\frac{2}{q^2}.$$

Note that the estimate (2) in [4, Ch. 25] is still valid under the weaker assumption that $\|\alpha-a/q\|\ll 1/q^2$. As in [4, Ch. 26], we have $P<q\leq Q$,

$$\int_0^1|S(\alpha)|^2\,d\alpha\ll N\log N,$$

and

$$S(-2\alpha)\ll N(\log N)^{-B/2+4}.$$

Hence the contribution of $\mathfrak{m}$ to $f(N)$ is

$$\int_{\mathfrak{m}}S(\alpha)^2S(-2\alpha)\,d\alpha\ll\left(\max_{\alpha\in\mathfrak{m}}|S(-2\alpha)|\right)\int_0^1|S(\alpha)|^2\,d\alpha\ll N^2(\log N)^{-B/2+5}.\tag{5}$$

Combining (4) and (5) and noting that $-B/2+5=-A$ we obtain

$$f(N)=\mathfrak{S}N^2+O(N^2(\log N)^{-A}).$$

We remark that the method may be adapted to estimate the number of solutions $(p_1, ..., p_k)$ to the linear equation $a_1p_1 + ... + a_kp_k = b$ with $p_1, ..., p_k \leq N$ for $k \geq 2$, where $a_1, ..., a_k \in \mathbb{Z} \setminus \{0\}$ are coprime and do not have the same sign. Naturally, we are led to considering

$$\int_0^1 S(a_1\alpha)S(a_2\alpha)\cdots S(a_k\alpha)e(-b\alpha)\,d\alpha.$$

One can define the major arc $\mathfrak{M}$ and the minor arc $\mathfrak{m}$ in the same way. The estimation of the contribution of $\mathfrak{M}$ is similar but more complicated, where the arithmetical features of $a_1, ..., a_k$ play a vital role in determining the main term, while the estimation of the contribution of $\mathfrak{m}$ needs slight modifications. Note that

$$\left|\int_{\mathfrak{m}} S(a_1\alpha)S(a_2\alpha)\cdots S(a_k\alpha)e(-b\alpha)\,d\alpha\right| \leq \prod_{j=1}^{k}\left(\int_{\mathfrak{m}} |S(a_j\alpha)|^k\,d\alpha\right)^{1/k}$$

by Hölder's inequality. For each $1 \leq j \leq k$, we have

$$\int_{\mathfrak{m}} |S(a_j\alpha)|^k\,d\alpha \leq \max_{\alpha\in\mathfrak{m}} |S(a_j\alpha)|^{k-2}\int_0^1 |S(a_j\alpha)|^2\,d\alpha = \max_{\alpha\in\mathfrak{m}} |S(a_j\alpha)|^{k-2}\int_0^1 |S(\alpha)|^2\,d\alpha.$$

From here we can proceed as before. However, the method just described may fail to work when the contribution of $\mathfrak{m}$ dominates. One such example is the twin prime problem ($k = 2$, $a_1 = -1$, $a_2 = 1$, and $b = 2$), in which case the above estimate for the contribution of $\mathfrak{m}$ dominates that of $\mathfrak{M}$.

References

[1] T. F. Bloom and O. Sisask, *Breaking the logarithmic barrier in Roth's theorem on arithmetic progressions*, preprint, 2020. arXiv:2007.03528.

[2] V. Brun, *La série 1/5+1/7+1/11+1/13+1/17+1/19+1/29+1/31+1/41+1/43+1/59+1/61+..., où les dénominateurs sont nombres premiers jumeaux est convergente ou finie*, Bull. Sci. Math. **43** (1919), 100–104, 124–128.

[3] J. G. van der Corput, *Über Summen von Primzahlen und Primzahlquadraten*, Math. Ann. **116** (1939), 1–50.

[4] H. Davenport, *Multiplicative Number Theory*, 3rd. ed., Grad. Texts in Math., vol. 74, Springer-Verlag, New York, 2000. Revised and with a preface by H. L. Montgomery.

[5] P. Erdős and P. Turán, *On some sequences of integers*, J. Lond. Math. Soc. **11** (1936), 261–264.

[6] B. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*, Ann. of Math. **167** (2008), 481–547.

[7] B. Green and T. Tao, *Linear equations in primes*, Ann. of Math. **171** (2) (2010), 1753–1850.

[8] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. **175** (2) (2012), 541–566.

[9] B. Green, T. Tao and T. Ziegler, *An inverse theorem for the Gowers $U^{s+1}[N]$-norms*, Ann. of Math. **176** (2) (2012), 1231–1372.

[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th. ed., Oxford Univ. Press, Oxford, 2008. Revised by D. R. Heath-Brown and J. H. Silverman; With a forward by A. J. Wiles.

[11] K. F. Roth, *On certain sets of integers*. J. Lond. Math. Soc. **28** (1953), 245–252.

[12] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.

Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA
Email address: steve.fan.gr@dartmouth.edu